QUASIPERIODIC SURFACE MARYLAND MODELS ON QUANTUM GRAPHS

KONSTANTIN PANKRASHKIN

ABSTRACT. We study quantum graphs corresponding to isotropic lattices with quasiperiodic coupling constants given by the same expressions as the coefficients of the discrete surface Maryland model. The absolutely continuous and the pure point spectra are described. It is shown that the transition between them is governed by the Hill operator corresponding to the edge potential.

1. INTRODUCTION

The present paper is devoted to the spectral analysis of a special class of quasiperiodic interactions on quantum graphs. We are going to show how some the theory of discrete quasiperiodic operators can be transferred to the quantum graph case using the tools of the operator theory.

The paper [9] provided the first explicit example of a difference quasiperiodic operator having a pure point spectrum dense everywhere; this operator is often referred to as the Maryland model. Later the class of such Hamiltonians was considerably extended in several directions, e.g. to the multidimensional case and to more general coefficients, see e.g. [1,7]. The papers [2,3,10] studied interactions similar to the Maryland model but supported by a subspace (surface Maryland model). In this case the quasiperiodic perturbation leaves unchanged the absolutely continuous spectrum of the unperturbed operator but produces a dense pure point spectrum of the rest of the real line.

On the other hand, discrete operators are closely related to the quantum graph models, i.e. differential operators acting on geometric configuration consisting of segments, see eg. [11,12]. The aim of the present paper is to provide an analog of the surface Maryland model for quantum graphs and to study its spectral properties. The work is a natural continuation of our previous paper [14] where we considered full-space Maryland quantum graph model.

2. THE MODEL OPERATOR

Let us describe first some basic constructions for quantum graphs. For a detailed discussion see e.g. [8,11,12]. There are many approaches to the study of the spectra of quantum graphs, we use the one from [5,13] based on the theory of self-adjoint extensions.

We consider a quantum graph whose set of vertices is identified with $\mathbb{Z}^d$, $d \geq 2$ (i.e. we explicitly need a multidimensional lattice). By $\mathbf{h}_j$, $j = 1, \dots, d$, we denote the standard basis vectors of $\mathbb{Z}^d$. Two vertices $\mathbf{m}$, $\mathbf{m}'$ are connected by an oriented edge $\mathbf{m} \to \mathbf{m}'$ iff $\mathbf{m}' = \mathbf{m} + \mathbf{h}_j$ for some $j \in \{1, \dots, d\}$; this edge is denoted as $(\mathbf{m}, j)$ and one says that $\mathbf{m}$ is the initial vertex and $\mathbf{m}' \equiv \mathbf{m} + \mathbf{h}_j$ is the terminal vertex.

Replace each edge $(\mathbf{m}, j)$ by a copy of the segment $[0, 1]$ in such a way that 0 is identified with $\mathbf{m}$ and 1 is identified with $\mathbf{m} + \mathbf{h}_j$. In this way we arrive at a certain topological set carrying a natural metric structure. The quantum state space of the

system is

$$\mathcal{H} := \bigoplus_{(\mathbf{m},j)\in\mathbb{Z}^d\times\{1,\dots,d\}} \mathcal{H}_{\mathbf{m},j}, \quad \mathcal{H}_{\mathbf{m},j} = \mathcal{L}^2[0,1],$$

and the vectors $f \in \mathcal{H}$ will be denoted as $f = (f_{\mathbf{m},j})$, $f_{\mathbf{m},j} \in \mathcal{H}_{\mathbf{m},j}$, $\mathbf{m} \in \mathbb{Z}^d$, $j = 1,\dots,d$.

Let us introduce a Schrödinger operator acting in $\mathcal{H}$. Fix a real-valued potential $q \in \mathcal{L}^2[0,1]$ and some real constants $\alpha(\mathbf{m})$, $\mathbf{m} \in \mathbb{Z}^d$. Set $A := \operatorname{diag}\big(\alpha(\mathbf{m})\big)$; this is a self-adjoint operator in $\ell^2(\mathbb{Z}^d)$. Denote by H_A the operator acting as

$$(f_{\mathbf{m},j}) \mapsto \Big(-f''_{\mathbf{m},j} + qf_{\mathbf{m},j}\Big), \tag{1a}$$

on functions $f = (f_{\mathbf{m},j}) \in \bigoplus_{\mathbf{m},j} H^2[0,1]$ satisfying the following boundary conditions:

$$f_{\mathbf{m},j}(0) = f_{\mathbf{m}-\mathbf{h}_k,k}(1) =: f(\mathbf{m}), \quad j,k = 1,\dots,d, \quad \mathbf{m} \in \mathbb{Z}^d, \tag{1b}$$

(which means the continuity at all vertices) and

$$f'(\mathbf{m}) = \alpha(\mathbf{m})f(\mathbf{m}), \quad \mathbf{m} \in \mathbb{Z}^d, \tag{1c}$$

where

$$f'(\mathbf{m}) := \sum_{j=1}^{d} f'_{\mathbf{m},j}(0) - \sum_{j=1}^{d} f'_{\mathbf{m}-h_j,j}(1). \tag{1d}$$

The constants $\alpha(\mathbf{m})$ are usually referred to as *Kirchhoff coupling constants* and interpreted as the strengths of zero-range impurity potentials placed at the corresponding vertices. The zero coupling constants correspond hence to the ideal couplings and are usually referred to as the standard boundary conditions.

We are going to study the above operator H_A for a special choice of the coefficients $\alpha(\mathbf{m})$. Namely, take $d_1 \in \{1,\dots,d-1\}$ and set $d_2 := d - d_1$. In what follows one represents any $\mathbf{m} \in \mathbb{Z}^d$ as $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2)$ with $\mathbf{m}_1 \in \mathbb{Z}^{d_1}$ and $\mathbf{m}_2 \in \mathbb{Z}^{d_2}$

Pick $g \neq 0$, $\boldsymbol{\omega} \in \mathbb{R}^{d_2}$, $\varphi \in \mathbb{R}$ with

$$\varphi \neq \boldsymbol{\omega}\mathbf{m}_2 \mod \frac{1}{2}, \quad \mathbf{m}_2 \in \mathbb{Z}^{d_2} \tag{2}$$

and set

$$\alpha(\mathbf{m}) := g \tan \pi\big(\boldsymbol{\omega}\mathbf{m}_2 + \varphi\big), \quad m \in \mathbb{Z}^d.$$

This operator will be noted simply by H.

To formulate the results we need some additional constructions. Denote by s and c the solutions to $-y'' + qy = zy$ satisfying $s(0;z) = c'(0;z) = 0$ and $s'(0;z) = c(0;z) = 1$, $z \in \mathbb{C}$, and set $\eta(z) := s(1;z) + c'(1;z)$. Consider an auxiliary one-dimensional Hill operator

$$L = -\frac{d^2}{dx^2} + Q, \quad Q(x+n) = q(x), \quad (x,n) \in [0,1) \times \mathbb{Z}. \tag{3}$$

It is known that $\operatorname{spec} L = \eta^{-1}\big([-2,2]\big)$.

Theorem 1. *For any $\boldsymbol{\omega}$ and φ one has $\operatorname{spec} L \subset \operatorname{spec} H$. If the components of $\boldsymbol{\omega}$ are rationally independent, then the spectrum of H in $\eta^{-1}\big((-2,2)\big)$ is purely absolutely continuous. If $\boldsymbol{\omega}$ satisfies additionally the Diophantine condition*

(4) *there are $C, \beta > 0$ with $|\boldsymbol{\omega}\mathbf{m}_2 - r| \geq C|\mathbf{m}_2|^{-\beta}$ for all $\mathbf{m}_2 \in \mathbb{Z}^{d_2} \setminus \{0\}$, $r \in \mathbb{Z}$,*

then the spectrum of H outside $\operatorname{spec} L$ is dense pure point.

The above theorem is a combination of propositions 3, 4, 8, 11 whose proof will be given below.

As easily seen, the location of the absolutely continuous spectrum of H is completely determined by the spectrum of the periodic operator L, and (under some additional assumptions) the rest of the spectrum is pure point. It is interesting to mention that a similar interlaced spectrum was found recently in a completely different model involving singular potentials [6].

3. Some construction from the theory of self-adjoint extensions

In this section we recall the operator-theoretical machinery which will be used to study the spectrum of H. For detailed discussion we refer to [5, Section 1].

Let S be a closed linear operator in a separable Hilbert space $\mathcal{H}$ with the domain $\operatorname{dom} S$. Assume that there exist an auxiliary Hilbert space $\mathcal{G}$ and two linear maps $\Gamma, \Gamma' : \operatorname{dom} S \to \mathcal{G}$ such that

- for any $f, g \in \operatorname{dom} S$ there holds $\langle f, Sg\rangle - \langle Sf, g\rangle = \langle \Gamma f, \Gamma' g\rangle - \langle \Gamma' f, \Gamma g\rangle$,
- the map $(\Gamma, \Gamma') : \operatorname{dom} S \to \mathcal{G} \oplus \mathcal{G}$ is surjective,
- the set $\ker(\Gamma, \Gamma')$ is dense in $\mathcal{H}$.

A triple $(\mathcal{G}, \Gamma, \Gamma')$ with the above properties is called a *boundary triple* for S. If S^* is symmetric, boundary triples deliver an effective description of all self-adjoint restrictions of S. If A is a self-adjoint operator in $\mathcal{G}$, then the restriction of S to the vectors f satisfying the abstract boundary conditions $\Gamma' f = A\Gamma f$ is a self-adjoint operator in $\mathcal{H}$, which we denote by H_A. Another example is the "distinguished" restriction H^0 corresponding to the boundary conditions $\Gamma f = 0$. One can show that a self-adjoint restriction H' of S can be represented as H_A with a suitable A iff $\operatorname{dom} H' \cap \operatorname{dom} H^0 = \operatorname{dom} S^*$; such restrictions are called *disjoint to* H^0. The resolvents of H^0 and H_A as well as their spectral properties are connected by Krein's resolvent formula, which will be described now.

Let $z \notin \operatorname{spec} H^0$. For $g \in \mathcal{G}$ denote by $\gamma(z)g$ the unique solution to the abstract boundary value problem $(S - z)f = 0$ with $\Gamma f = g$. Clearly, $\gamma(z)$ is a linear map from $\mathcal{G}$ to $\mathcal{H}$ and an isomorphism between $\mathcal{G}$ and $\ker(S-z)$; it is sometimes referred to as the Krein γ-field associated with the boundary triple. Denote also by $M(z)$ the bounded linear operator on $\mathcal{G}$ given by $M(z)g = \Gamma'\gamma(z)g$; this operator will be referred to as the Weyl function (or the abstract Dirichlet-to-Neumann map) corresponding to the boundary triple $(\mathcal{G}, \Gamma, \Gamma')$. The operator-valued functions γ and M are analytic outside $\operatorname{spec} H^0$, and $M(z)$ is self-adjoint for real z. If these maps are known, one can relate the operators H_A and H^0 as follows,:

Proposition 2. *For $z \notin \operatorname{spec} H^0 \cup \operatorname{spec} H_A$ the operator $M(z) - A$ acting on $\mathcal{G}$ has a bounded inverse defined everywhere, and*

$$(H_A - z)^{-1} = (H^0 - z)^{-1} - \gamma(z)\big(M(z) - A\big)^{-1}\gamma(\bar z)^*. \tag{5}$$

In particular, the set $\operatorname{spec} H_A \setminus \operatorname{spec} H^0$ consists exactly of $z \in \mathbb{R} \setminus \operatorname{spec} H^0$ such that $0 \in \operatorname{spec}\big(M(z) - A\big)$. The same correspondence holds for the eigenvalues, i.e. $z \in \mathbb{R} \setminus \operatorname{spec} H^0$ is an eigenvalue of H_A iff 0 is an eigenvalue of $M(z) - A$, and $\gamma(z)$ is an isomorphism of the corresponding eigensubspaces.

The maps γ and M satisfy a number of important properties. In particular, γ and M depend analytically on their argument (outside of $\operatorname{spec} H^0$), $M(z)$ satisfies $M(\bar z) = M(z)^*$ and

$$\text{for any non-real } z \text{ there is } c_z > 0 \text{ with} \frac{\Im M(z)}{\Im z} \geq c_z, \text{ and} \tag{6}$$

$$M'(\lambda) = \gamma(\lambda)^*\gamma(\lambda) > 0 \text{ for } \lambda \in \mathbb{R} \setminus \operatorname{spec} H^0. \tag{7}$$

Furthermore,

$$\gamma(z)^* f = 0 \text{ for any } f \in \ker(S-z)^\perp \equiv \operatorname{ran}\gamma(z)^\perp, \tag{8}$$

see [5, Section 1] for more details.

Below it will be useful to have a certain relationship between the resolvent of H_A and that of the operator H_0 (i.e. H_A with $A=0$). Clearly, $(\mathcal{G},\widetilde{\Gamma},\widetilde{\Gamma}')$ with $\widetilde{\Gamma}:=\Gamma'$ and $\widetilde{\Gamma}':=-\Gamma$ is a new boundary triple for S. With respect to this boundary triple H_0 is the distinguished extension, and one can easily calculate (at least for non-real z) the corresponding Krein γ-field $\widetilde{\gamma}(z):=\gamma(z)M(z)^{-1}$ and the Weyl function $\widetilde{M}(z):=-M(z)^{-1}$ which then extend by the analyticity to $\mathbb{R}\setminus\operatorname{spec} H_0$. The operator H_A corresponds now to the boundary conditions: $\widetilde{\Gamma}f=0$ for $\Gamma f\in\ker A$ and $\widetilde{\Gamma}'f=-A^{-1}\widetilde{\Gamma}f$ otherwise. Hence, if the operator A is not invertible, one cannot use the above proposition 2 for the resolvents. One can avoid this difficulty as follows. Denote $\mathcal{G}':=\ker A^\perp$; clearly, $\mathcal{G}'$ is a closed linear subspace of $\mathcal{G}$. Denote by P the orthogonal projection from $\mathcal{G}$ to $\mathcal{G}'$ and set $\Pi:=P\widetilde{\Gamma}P$ and $\Pi':=P\widetilde{\Gamma}'P$ and $S':=S|_{\widetilde{\Gamma}^{-1}(\mathcal{G}')}$, then $(\mathcal{G}',\Pi,\Pi')$ is a boundary triple for S' with the Krein field $\nu(z):=P\widetilde{\gamma}(z)P$ and the Weyl function $N(z)=P\widetilde{M}(z)P$, and H_A and H_0 become disjoint self-adjoint restrictions of S'. Hence, one has the following resolvent formula

$$(H_0-z)^{-1}-(H_A-z)^{-1}=\nu(z)\big(N(z)-B\big)\nu^*(\bar z),\quad B=-PA^{-1}P, \tag{9}$$

or

$$\begin{aligned}(H_0-z)^{-1}&-(H_A-z)^{-1}\\ &=P\gamma(z)M(z)^{-1}\big(PA^{-1}P-PM(z)^{-1}P\big)M(z)^{-1}P\gamma^*(\bar z).\end{aligned} \tag{10}$$

4. Resolvents for the quantum graph

We are going to show now how the constructions of section 3 apply to the quantum graph Hamiltonian H (see Section 2).

Denote by S the operator acting as (1a) on the functions f satisfying only the boundary conditions (1b). On the domain of S one can define linear maps

$$f\mapsto\Gamma f:=\big(f(\mathbf{m})\big)_{\mathbf{m}\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d),\quad f\mapsto\Gamma' f:=\big(f'(\mathbf{m})\big)_{\mathbf{m}\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d).$$

One can show that $(\mathbb{Z}^d,\Gamma,\Gamma')$ form a boundary triple for S. The distinguished restriction to $\ker\Gamma$, H^0, acts as (1a) on functions $(f_{\mathbf{m},j})$ with $f_{\mathbf{m},j}\in H^2[0,1]$ satisfying the Dirichlet boundary conditions, $f_{\mathbf{m},j}(0)=f_{\mathbf{m},j}(1)=0$ for all m,j, and the spectrum of H^0 is just the Dirichlet spectrum of $-\dfrac{d^2}{dt^2}+q$ on the segments $[0,1]$; we will refer to $\operatorname{spec} H^0$ as to the *Dirichlet spectrum* of the graph.

Let us construct the maps $\gamma(z)$ and $M(z)$ for the above boundary triple. In in terms of these functions s and c one has obviously

$$\begin{aligned}\big(\gamma(z)\xi\big)_{\mathbf{m},j}(t)=\frac{1}{s(1;z)}\Big(&\xi(\mathbf{m}+\mathbf{h}_j)s(t;z)\\ &+\xi(\mathbf{m})\big(s(1;z)c(t;z)-c(1;z)s_j(t;z)\big)\Big),\\ &t\in[0,1],\quad(\mathbf{m},j)\in\mathbb{Z}^d\times\{1,\dots,d\}.\end{aligned}$$

The corresponding Weyl function $M(z):\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)$ is given by

$$M(z)\xi(\mathbf{m})=\frac{1}{s(1;z)}\sum_{j=1}^d\cdot\Big(\xi(\mathbf{m}-\mathbf{h}_j)+\xi(\mathbf{m}+\mathbf{h}_j)-\eta(z)\,\xi(\mathbf{m})\Big),\quad\xi\in\ell^2(\mathbb{Z}^d),$$

where $\eta(z) := c(1;z) + s'(1;z)$ is the Hill discriminant associated with the potential q. It is useful to introduce the discrete Hamiltonian Δ_d in $\ell^2(\mathbb{Z}^d)$ by

$$\Delta_d \xi(\mathbf{m}) = \sum_{\mathbf{m}':|\mathbf{m}-\mathbf{m}'|=1} \xi(\mathbf{m}'),$$

then one has obviously

$$M(z) := a(z)\big(\Delta_d - d\eta(z)\big), \quad a(z) := \frac{1}{s(1;z)}. \tag{11}$$

By proposition 2, outside of the discrete set $\operatorname{spec} H^0$, the spectrum of H consists of the real z satisfying $0 \in \operatorname{spec}\big(M(z) - A\big)$. It is important to emphasize that, for real z, the operator $M(z) - A$ is exactly the surface Maryland model studied in [3]. The results of [2] imply that

- the operator $M(z) - A$ has no bounded inverse if $\big|\eta(z)\big| \le 2$,
- if the components of $\boldsymbol{\omega}$ are rationally independent, then the spectrum in the interval $a(z)\big(2d - \eta(z), 2d + \eta(z)\big)$ is purely absolutely continuous,
- for Diophantine $\boldsymbol{\omega}$ the rest of the real line is covered by the dense pure point spectrum.

Using first of these properties and proposition 2 one immediately obtains

Proposition 3. $\operatorname{spec} L \subset \operatorname{spec} H$.

(It is sufficient to recall that the set $\big|\eta(z)\big| \le 2$ coincides with spectrum of L.) Nevertheless, one is not able to conclude about the spectral nature of H from that of $M(z) - A$ using just the general theory of self-adjoint extensions [4]. We are going to use some additional considerations from [2, 7] in order to understand completely the spectral properties of H.

We will also use the formula (9) relating the resolvent of H_A and H_0. Denoting by P the orthogonal projection from $\ell^2(\mathbb{Z}^d)$ to $\ell^2(\mathbb{Z}^{d_2})$ on obtains $N(z) = -PM(z)P$. The parameter operator $B := -(PAP)^{-1}$ is the multiplication by $-g^{-1}\cot\pi(\boldsymbol{\omega}\mathbf{m}_2 + \varphi) \equiv g^{-1}\tan\pi(\boldsymbol{\omega}\mathbf{m}_2 + \varphi + 1/2)$. It is also important to emphasize that, as shown in [13], one has $\operatorname{spec} H_0 = \operatorname{spec} L \cup \operatorname{spec} H^0$ where L is the one-dimensional Hill operator (3), and each point of $\operatorname{spec} H^0$ is an infinitely degenerate eigenvalue.

5. The absolutely continuous spectrum

Below we will use actively the Fourier transform. Denote $\mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\}$ and $\mathbb{T}^n := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}} \subset \mathbb{C}^n$. For $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \subset \mathbb{C}^n$ and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$ we write $\boldsymbol{\theta}^{\mathbf{p}} := \theta_1^{p_1} \dots \theta_n^{p_n}$, and in this context $k \in \mathbb{Z}$ will be identified with the vector $(k, \dots, k) \in \mathbb{Z}^n$, i.e. $\boldsymbol{\theta}^{-1} := \theta_1^{-1} \dots \theta_l^{-1}$ etc. We denote by F_n the Fourier transform carrying $\ell^2(\mathbb{Z}^n)$ to $\mathcal{L}^2(\mathbb{T}^n)$,

$$F_n\psi(\boldsymbol{\theta}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} \psi(\mathbf{n})\boldsymbol{\theta}^{\mathbf{n}}, \quad F_n^{-1} f(\mathbf{n}) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} f(\boldsymbol{\theta})\boldsymbol{\theta}^{-\mathbf{n}-1} d\boldsymbol{\theta}.$$

Each $\boldsymbol{\theta} \in \mathbb{T}^d$ will be represented $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ with $\boldsymbol{\theta}_1 \in \mathbb{T}^{d_1}$ and $\boldsymbol{\theta}_2 \in \mathbb{T}^{d_2}$.

We will repeat first some algebraic manipulations in the spirit of [3]. Without loss of generality assume $g > 0$ (otherwise one can change the signs of $\boldsymbol{\omega}$ and φ).

Consider the operator $L(z) := M(z) - A = M(z) + PvP$, where v is an operator in $\ell^2(\mathbb{Z}^{d_2})$ acting as $vf(\mathbf{m}_2) = -g\tan(\boldsymbol{\omega}\mathbf{m}_2 + \varphi)f(\mathbf{m}_2)$, $\mathbf{m}_2 \in \mathbb{Z}^{d_2}$. For $\Im z \ne 0$ the operator $M(z)$ is invertible (as its imaginary part is non-degenerate) and one has

$$L(z)^{-1} = M(z)^{-1} - M(z)^{-1}T(z)M(z)^{-1}, \quad T(z) = v - T(z)M(z)^{-1}v.$$

Obviously one can write $T(z) = Pt(z)P$ where the operator $t(z)$ acting in $\ell^2(\mathbb{Z}^{d_2})$ satisfies $t(z) = v + t(z)N(z)v$. Formally one has $t(z) = v(1 - N(z)v)^{-1}$, and it is needed to show that the operator in question is really invertible.

Let U be the unitary operator in $\ell^2(\mathbb{Z}^{d_2})$ defined by the relation

$$(uf)(\mathbf{m}_2) = e^{-2\pi i\boldsymbol{\omega}\mathbf{m}_2} f(\mathbf{m}_2),$$

then, denoting $\chi := e^{-2\pi i\varphi}$, one can write

$$v = -\frac{g}{i}\frac{1-\chi U}{1+\chi U}.$$

As $\Im N(z) \geq 0$ for $\Im z \geq 0$, the operator $i + gN(z)$ is invertible for such z. Hence, for $\Im z \geq 0$ after a simple algebra one obtains

$$1 - N(z)v = \big(gN(z)+i\big)\big(1-b(z)\chi U\big)\big(i(1+\chi U)\big)^{-1}$$

where $b(z) = \big(gN(z)-i\big)\big(i+gN(z)\big)^{-1}$. In order to represent the inverse operator in terms of the Neumann series it is sufficient to show that $|b(z)| < 1$ for some z. To see this, it is useful to pass to the Fourier representation.

For $\lambda \in \mathbb{C}$ denote $G_d(\lambda) := (\Delta_d - \lambda)^{-1}$. Recall that in the Fourier representation Δ_d becomes the multiplication by the function $\Delta_d(\boldsymbol{\theta}) = \sum_j(\theta_j + \theta_j^{-1})$, hence the matrix of $G_d(\lambda)$ is given by

$$G_d(\mathbf{m}-\mathbf{m}';\lambda) = \frac{1}{(2\pi i)^d}\int_{\mathbb{T}^d}\frac{\theta^{-(\mathbf{m}-\mathbf{m}')-1}d\boldsymbol{\theta}}{\Delta_d(\boldsymbol{\theta})-\lambda}.$$

On the other hand, the matrix of the operator $N(z)$ is $N(\mathbf{m}_2 - \mathbf{m}_2'; z) = -a(z)^{-1}G_d\big((0,\mathbf{m}_2)-(0,\mathbf{m}_2'); d\eta(z)\big)$, hence

$$\begin{aligned}N(\mathbf{m}_2-\mathbf{m}_2';z) &= -a(z)^{-1}\frac{1}{(2\pi i)^d}\int_{\mathbb{T}^{d_2}}\boldsymbol{\theta}_2^{-(\mathbf{m}_2-\mathbf{m}_2')-1}d\theta_2\int_{\mathbb{T}^{d_1}}\frac{\boldsymbol{\theta}_1^{-1}d\boldsymbol{\theta}_1}{\Delta_d(\boldsymbol{\theta})-d\eta(z)}\\ &= -a(z)^{-1}\frac{1}{(2\pi i)^{d_2}}\int_{\mathbb{T}^{d_2}}G_{d_1}\big(\mathbf{0};d\eta(z)-\Delta_{d_2}(\boldsymbol{\theta}_2)\big)\boldsymbol{\theta}_2^{-(\mathbf{m}_2-\mathbf{m}_2')-1}d\boldsymbol{\theta}_2.\end{aligned} \tag{12}$$

In particular, it is clear that in the Fourier representation $N(z)$ is the multiplication by the function

$$N(\boldsymbol{\theta}_2;z) = -a(z)^{-1}G_{d_1}\big(\mathbf{0};d\eta(z)-\Delta_{d_2}(\boldsymbol{\theta}_2)\big). \tag{13}$$

As $\Im N(z) > 0$ for $\Im z > 0$, the imaginary part $\Im N(\boldsymbol{\theta}_2;z)$ is positive for such z. The operator $b(z)$ in the Fourier representation becomes the multiplication by the function

$$b(\boldsymbol{\theta}_2,z) = \frac{gN(\boldsymbol{\theta}_2,z)-i}{gN(\boldsymbol{\theta}_2,z)+i},$$

hence $\|b(z)\| \equiv \sup_{\boldsymbol{\theta}_2}|b(\boldsymbol{\theta}_2,z)| < 1$ for $\Im z > 0$. Therefore, one can represent

$$\begin{aligned}t(z) &= v(1-N(z)v)^{-1}\\ &= -g(1-\chi U)\big(1-b(z)\chi U\big)^{-1}\big(gN(z)+i\big)^{-1}\\ &= -g(1-\chi U)\sum_{m=0}^{\infty}\chi^m\big(b(z)U\big)^m\\ &= -g\big(gN(z)+i\big)^{-1}\Big(1-2i\sum_{m=1}^{\infty}\big(gN(z)+i\big)^{-1}U\big(b(z)U\big)^{m-1}\Big).\end{aligned} \tag{14}$$

and one has

$$\big(M(z)-A\big)^{-1} = M(z)^{-1} - M(z)^{-1}Pt(z)PM(z)^{-1}.$$

After these preparations we can prove

Proposition 4. *Denote $I := \eta^{-1}\big((-2,2)\big)$. If the vector $\boldsymbol{\omega}$ has rationally independent components, then the operator H has only absolutely continuous spectrum in I.*

Proof. According to the general spectral theory we need to show that there exists a dense subset $\mathcal{L}$ of $\mathcal{H}$ such that the limit $\lim_{\varepsilon\to 0+}\Im\langle f,(H-\lambda-i\varepsilon)^{-1}f\rangle$ exists and is finite for all $g\in\mathcal{L}$ and $\lambda\in I$.

Represent

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1,\quad \mathcal{H}_0:=\Big(\bigcup_{\Im z\neq 0}\gamma(z)\big(\ell^2(\mathbb{Z}^d)\big)\Big)^{\perp},\quad \mathcal{H}_1:=\mathcal{H}_0^{\perp}; \tag{15}$$

in other words, $\mathcal{H}_1$ is the closure of the linear hull of the set $\{\gamma(z)\varphi:\,\Im z\neq 0,\,\varphi\in\ell^2(\mathbb{Z}^d)\}$.

By the Krein resolvent formula, for any $f\in\mathcal{H}_0$ and any z with $\Im z\neq 0$ one has $\gamma^*(z)f=0$. Hence, by (5), there holds $(H-z)^{-1}f=(H^0-z)^{-1}f$, hence $\lim_{\varepsilon\to 0+}\Im\langle f,(H-\lambda-i\varepsilon)^{-1}f\rangle=0$n because $(H^0-\lambda)^{-1}$ is a bounded self-adjoint operator.

Consider the vectors $f=\gamma(\zeta)h$ for $h=\big(M(\zeta)-A\big)^{-1}\xi$, $\Im\zeta\neq 0$. These vectors form a dense subset in $\mathcal{H}_1$ as ξ runs over a dense subset of $\ell^2(\mathbb{Z}^d)$. By elementary calculations (see e.g. section 3 in [5]) one can write

$$(H-\lambda-i\varepsilon)^{-1}f=\frac{1}{\zeta-\lambda-i\varepsilon}\Big(f-\gamma(\lambda+i\varepsilon)\big(M(\lambda+i\varepsilon)-A\big)^{-1}\xi\Big).$$

Hence it is sufficient to show that $\lim_{\varepsilon\to 0+}\Im\langle\delta_{\mathbf{m}},(M(\lambda+i\varepsilon)-A)^{-1}\delta_{\mathbf{m}}\rangle$ exists and is finite for any $\mathbf{m}\in\mathbb{Z}^d$. In view of the series representation for $(M(z)-A)^{-1}$ it is sufficient to show that the series converges for real $z\in I$ and not only for $\Im z>0$. On the other hand, $(M(z)-A)^{-1}=a(z)^{-1}(\Delta_d-d\eta(z)-a(z)^{-1}A)^{-1}$, and it is shown in [2, Theorem 3.1] that $\lim_{\varepsilon\to 0+}\Im\langle\delta_{\mathbf{m}},(\Delta_d-b-cA)^{-1}\delta_{\mathbf{m}}\rangle$ exists and is finite for any $b\in(-2d,2d)$ and any $c\in\mathbb{R}$. This completes the proof. $\square$

6. The pure point spectrum

In this section we will use the second version of the resolvent formula, Eq. (9). Hence for $z\notin\operatorname{spec}H_0$ we have the equivalence $z\in\operatorname{spec}H$ iff $0\in\operatorname{spec}(N(z)-B)$. Here N is a translationally invariant operator in $\ell^2(\mathbb{Z}^{d_2})$ whose matrix elements are given by (12), and the operator B, as already mentioned below, in the multiplication by the sequence $g^{-1}\tan\pi(\boldsymbol{\omega}\mathbf{m}_2+\varphi+1/2)$. It is useful to set $g'=-g$, $\boldsymbol{\omega'}:=-\boldsymbol{\omega}$, $\varphi':=-\varphi-1/2$, then B becomes a multiplication by $-g'\tan\pi(\boldsymbol{\omega'}\mathbf{m}_2+\varphi')$ with $g'>0$.

To alleviate the notation, below we will write d instead of d_2 and drop the indices for g', $\boldsymbol{\omega'}$ and φ' as this does not lead to confusions.

Introduce the operators

$$D(z):=\big(N(z)-ig\big)^{-1},\quad C(z):=-\big(N(z)+ig\big)\big(N(z)-ig\big)^{-1};$$

they are defined at least for z with $\Re z\notin\operatorname{spec}H_0$ and $|\Im z|$ sufficiently small. One can write for such z the identity

$$N(z)-B=D(z)^{-1}\big(1-\chi C(z)U\big)(1+\chi U)^{-1}. \tag{16}$$

Recall that under the Fourier transform $N(z)$ becomes the multiplication by the function $N(\boldsymbol{\theta},z)$ given by (13), the operators $D(z)$ and $C(z)$ become the multiplications by $D(\boldsymbol{\theta},z):=\big(N(\boldsymbol{\theta},z)-ig\big)^{-1}$ by $C(\boldsymbol{\theta},z):=-\big(N(\boldsymbol{\theta},z)+ig\big)\big(N(\theta,z)-ig\big)^{-1}$, respectively, and U becomes a shift operator, $Uk(\boldsymbol{\theta})=k(e^{-2\pi i\omega_1}\theta_1,\ldots,e^{-2\pi i\omega_d}\theta_d)$.

Consider an arbitrary segment $[a,b]\subset\mathbb{R}\setminus\operatorname{spec}H_0$. Recall that the spectrum of H_0 coincides with the spectrum of L up to the discrete set $\operatorname{spec}H^0$. Eq. (6), the

analyticity of γ, and the self-adjointness of $N(z)$ for real z imply the existence of $\delta' > 0$ such that $\|\Im N(z)\| \leq g/2$ for $z \in Z := \{z \in \mathbb{C} : |\Im z| \leq \delta',\ \Re z \in [a,b]\}$. At the same time, this means that $|\Im N(\boldsymbol{\theta}, z)| \leq g/2$ for $z \in Z$. As follows from the integral representation, $N(\boldsymbol{\theta}, z)$ can be continued to an analytic function in $Z \times \Theta$, $\Theta := \{\boldsymbol{\theta} \subset \mathbb{C}^d : r < |\theta_j| < R\}$, $0 < r < 1 < R < \infty$. Choosing r and R sufficiently close to 1 one immediately sees that the function

$$C(\boldsymbol{\theta}, z) := \frac{g^2 - \big(\Im N(\boldsymbol{\theta}, z)\big)^2 - \big(\Re N(\boldsymbol{\theta}, z)\big)^2 - 2ig\Re N(\boldsymbol{\theta}, z)}{\big|N(\boldsymbol{\theta}, z) - ig\big|^2}$$

does not take values in $(-\infty, 0)$ for $(\boldsymbol{\theta}, z) \in \Theta \times Z$. Therefore, the function $f(\boldsymbol{\theta}, z) := \log C(\boldsymbol{\theta}, z)$ is well-defined and analytic in $\Theta \times Z$, where log denotes the principal branch of the logarithm. The Diophantine property (4) implies (see [7, Lemma 3.2]) that the operator $1 - U$ is a bijection on the set of functions v analytic in Θ with

$$\int_{\mathbb{T}^d} v(\boldsymbol{\theta})\boldsymbol{\theta}^{-1} d\boldsymbol{\theta} = 0.$$

Hence the function $t(\boldsymbol{\theta}, z) := (1-U)^{-1}\big(f(\boldsymbol{\theta}, z) - f_0(z)\big)$ is well-defined and analytic in $Z \times \Theta$, where

$$f_0(z) := \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} f(\boldsymbol{\theta}, z)\boldsymbol{\theta}^{-1} d\boldsymbol{\theta}. \tag{17}$$

Lemma 5. *The function f_0 is analytic in Z,*

$$\Re f_0(z) < 0 \quad \textit{for} \quad \Im z > 0, \tag{18}$$

$$\Re f(\boldsymbol{\theta}, z) = \Re t(\boldsymbol{\theta}, z) = \Re f_0(z) = 0 \quad \textit{for} \quad \Im z = 0. \tag{19}$$

For real λ one has $f_0(\lambda) = 2i\sigma(\lambda)$, where

$$\sigma(\lambda) = \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} \arctan \frac{N(\boldsymbol{\theta}, \lambda)}{g}\, \boldsymbol{\theta}^{-1} d\boldsymbol{\theta}.$$

The function σ is real-valued, strictly increasing, and continuously differentiable on $[a, b]$.

Proof. The analyticity of f_0 follows from its integral representation. Eq. (18) follows from (17) if one takes into account the inequalities $\Im N(\boldsymbol{\theta}, z) > 0$ for $\Im z > 0$ and $\Re \log z < 0$ for $|z| < 1$. Equalities (18) follows from from (17) and the real-valuedness of $N(\boldsymbol{\theta}, z)$ for real z.

By elementary calculations, for $x \in \mathbb{R}$ and $y > 0$ one has

$$g_1(x) := \frac{1}{2i} \log \frac{iy + x}{iy - x} \equiv \arctan \frac{x}{y} =: g_2(x). \tag{20}$$

In fact, this follows from

$$g_1'(x) = g_2'(x) = \frac{y}{x^2 + y^2} \tag{21}$$

and $g_1(0) = g_2(0) = 0$. Eq. (20) obviously implies $f_0(\lambda) = 2i\sigma(\lambda)$ for $\lambda \in \mathbb{R}$. Furthermore, as follows from (21),

$$\sigma'(\lambda) = \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} \frac{gN_\lambda'(\boldsymbol{\theta}, \lambda)}{N(\boldsymbol{\theta}, \lambda)^2 + g^2}\, \boldsymbol{\theta}^{-1} d\boldsymbol{\theta},$$

and, by (7), $\sigma'(\lambda) > 0$. □

An immediate corollary of the analyticity of f_0 and of (18) is

Lemma 6. *There exists $\varepsilon_0 > 0$ such that $\big|e^{f_0(\lambda)}\xi - 1\big| \leq 2\big|e^{f_0(\lambda + i\varepsilon)}\xi - 1\big|$ for all $\xi \in \mathbb{S}^1$, $\lambda \in [a, b]$, and $\varepsilon \in [0, \varepsilon_0]$.*

Denote by $t(z)$ and $f(z)$ the multiplication operators by $t(\boldsymbol{\theta},z)$ and $f(\boldsymbol{\theta},z)$ in $\mathcal{L}^2(\mathbb{T}^d)$, respectively. By definition of $t(\boldsymbol{\theta},z)$ for any
$xi \in \mathcal{L}^2(\mathbb{T}^d)$

$$\begin{aligned} &e^{t(z)}e^{f_0(z)}Ue^{-t(z)}\xi(\boldsymbol{\theta}) \\ &\quad= e^{t(\boldsymbol{\theta},z)}e^{f_0(\boldsymbol{\theta},z)}\exp\bigl(-t(z,e^{-2\pi i\omega_1}\theta_1,\ldots,e^{-2\pi i\omega_d}\theta_d)\bigr)U\varphi(\boldsymbol{\theta}) \\ &\quad= \exp\bigl(t(\boldsymbol{\theta},z)-Ut(\boldsymbol{\theta},z)+f_0(\boldsymbol{\theta},z)\bigr)U\varphi(\boldsymbol{\theta},z)=e^{f(z)}U\varphi(\boldsymbol{\theta})=C(z)U\varphi(\boldsymbol{\theta}). \end{aligned} \tag{22}$$

Therefore, one can rewrite Eq. (16) as

$$N(z)-B=D(z)^{-1}e^{t(z)}\bigl(1-e^{f_0(z)}\chi U\bigr)e^{-t(z)}\bigl(1+\chi U\bigr)^{-1}. \tag{23}$$

Proposition 7. *The set of the eigenvalues of H in $[a,b]$ is dense and coincides with the set of solutions λ to*

$$\sigma(\lambda)=\pi(\boldsymbol{\omega}\mathbf{m}+\varphi) \mod \pi, \quad \mathbf{m}\in\mathbb{Z}^d. \tag{24}$$

Each of these eigenvalues is simple, and for any fixed $\mathbf{m}\in\mathbb{Z}^d$ Eq. (24) has at most one solution $\lambda(\mathbf{m})$, and $\lambda(\mathbf{m})\neq\lambda(\mathbf{m}')$ for $\mathbf{m}\neq\mathbf{m}'$.

Proof. As follows from proposition 2 and the resolvent formula (9), the eigenvalues λ of H outside $\operatorname{spec} H_0$ are determined by the condition $\ker\bigl(N(\lambda)-B\bigr)\neq 0$, an their multiplicity coincides with the dimension of the corresponding kernels. Eq. (22) shows that the condition $(N(\lambda)-B)u=0$ is equivalent to $\bigl(1-e^{f_0(\lambda)}\chi U\bigr)e^{-t(\lambda)}\bigl(1+\chi U\bigr)^{-1}u=0$ or, denoting $v:=e^{-t(\lambda)}\bigl(1+\chi U\bigr)^{-1}u$, $(1-e^{f_0(\lambda)}\chi U\bigr)v=0$, which can be rewritten as

$$\chi U v=e^{-f_0(\lambda)}v, \quad v\neq 0. \tag{25}$$

As χU has the simple eigenvalues $e^{-2\pi i(\boldsymbol{\omega}\mathbf{m}+\varphi)}$, $\mathbf{m}\in\mathbb{Z}^d$, and the corresponding eigenvectors form a basis, Eq. (25) implies (24) if one takes into account the identity $f_0(\lambda)=2i\sigma(\lambda)$ proved in lemma 5. The rest follows from the monotonicity of σ, the inclusion $\operatorname{ran}\sigma\subset(-\pi/2,\pi/2)$, and the arithmetic properties (2) and (4). $\square$

As $[a,b]$ was an arbitrary interval from $\mathbb{R}\setminus\operatorname{spec} H_0$, one has an immediate corollary

Proposition 8. *The pure point spectrum of H is dense in $\mathbb{R}\setminus\operatorname{spec} H_0$.*

Now it remains to show that the spectrum of H in the interval considered is pure point.

Take some $\alpha>0$. For any $\delta>0$ we denote

$$\mathbb{S}^1_\delta=\bigcup_{m\in\mathbb{Z}^d}\Bigl\{\xi\in\mathbb{S}^1:|\operatorname{Arg}\xi-\operatorname{Arg}e^{2\pi i\boldsymbol{\omega}\mathbf{m}}|\le\delta\bigl(1+|\mathbf{m}|\bigr)^{-d-\alpha}\Bigr\}, \quad \widetilde{\mathbb{S}}^1_\delta:=\mathbb{S}^1\setminus\mathbb{S}^1_\delta.$$

Clearly, there holds

$$|1-\xi e^{-2\pi i\boldsymbol{\omega}\mathbf{m}}|\ge 2\pi^{-1}\delta\bigl(1+|\mathbf{m}|\bigr)^{-d-\alpha}, \quad \xi\in\widetilde{\mathbb{S}}^1_\delta, \quad m\in\mathbb{Z}^d. \tag{26}$$

Let $\Delta\subset[a,b]$ be an interval whose ends are not eigenvalues of H. Consider the mapping $h:\lambda\mapsto\chi e^{f_0(\lambda)}$. By lemma 5, h is a diffeomorphism between Δ and $h(\Delta)$. By proposition 7 one has $h\bigl(\lambda(\mathbf{m})\bigr)=e^{2\pi i\boldsymbol{\omega}\mathbf{m}}$. Take an arbitrary $\delta>0$ and denote

$$\Delta_\delta:=\Delta\cap h^{-1}(\mathbb{S}^1_\delta), \quad \widetilde{\Delta}_\delta:=\Delta\cap h^{-1}(\widetilde{\mathbb{S}}^1_\delta)\equiv\Delta\setminus\Delta_\delta.$$

Clearly, Δ_δ is a countable union of intervals, and the limit set $\bigcap_{\delta>0}\Delta_\delta$ coincides with the set of all the eigenvalues $\bigcup_m\{\lambda(m)\}$.

Lemma 9. *There exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ and any $\mathbf{n} \in \mathbb{Z}^d$ there exists $C > 0$ such that*

$$\big\|\big(N(\lambda + i\varepsilon) - B\big)^{-1}\delta_{\mathbf{n}}\big\| \le C \tag{27}$$

for all $\lambda \in \widetilde{\Delta}_\delta$, and $\varepsilon \in (0, \varepsilon_0)$.

Proof. Rewrite Eq. (23) in the form

$$\big(N(z) - B\big)^{-1} = \big(1 + \chi U\big)e^{t(z)}\big(1 - e^{f_0(z)}\chi U\big)^{-1}e^{-t(z)}D(z).$$

Note that the Fourier transform of $\delta_{\mathbf{n}}$ is the function $\boldsymbol{\theta} \mapsto \boldsymbol{\theta}^{\mathbf{n}}$. Denote $\Psi(z;\boldsymbol{\theta}) := e^{-t(\boldsymbol{\theta},z)}B(\boldsymbol{\theta}, z)\boldsymbol{\theta}^{\mathbf{n}}$. Due to the analyticity one can estimate uniformly in Z:

$$|\psi_z(\mathbf{m})| \le C' e^{-\rho|\mathbf{m}|}, \quad C', \rho > 0, \quad \psi_z := F_d^{-1}\Psi, \quad \|(1 + \chi U)e^{t(z)}\| \le C'.$$

Therefore, (27) follows from the inequality

$$\big\|(1 - e^{f_0(\lambda+i\varepsilon)}\chi U)^{-1}\Psi\big\| \le C. \tag{28}$$

Assume that ε_0 satisfies the conditions of lemma 6, then uniformly for $\lambda \in \Delta$ and $\varepsilon \in (0, \varepsilon_0)$ one has

$$\begin{aligned}\big|\big(F_d^{-1}(1 - e^{f_0(\lambda+i\varepsilon)}\chi U)^{-1}\Psi\big)(\mathbf{m})\big| &= \big|(1 - e^{f_0(\lambda+i\varepsilon)}\chi e^{2\pi i\boldsymbol{\omega}\mathbf{m}})^{-1}\psi_{\lambda+i\varepsilon}(\mathbf{m})\big| \\ &\le 2\big|(1 - e^{f_0(\lambda)}\chi e^{2\pi i\boldsymbol{\omega}\mathbf{m}})^{-1}\big| \cdot |\psi_{\lambda+i\varepsilon}(\mathbf{m})|.\end{aligned}$$

As in our case $h(\lambda) \equiv \chi e^{f_0(\lambda)} \in \widetilde{\mathbb{S}}^1_\delta$, due to (26) we have

$$\big|(1 - e^{f_0(\lambda)}\chi e^{-2\pi i\boldsymbol{\omega}\mathbf{m}})^{-1}\big| \le \frac{\pi}{2\delta}\big(1 + |\mathbf{m}|\big)^{d+\alpha}.$$

Finally,

$$\begin{aligned}\big\|(1 - e^{f_0(\lambda+i\varepsilon)}\chi U)^{-1}\Psi\big\|^2 &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\big|\big(F_d^{-1}(1 - e^{f_0(\lambda+i\varepsilon)}\chi U)^{-1}\Psi\big)(\mathbf{m})\big|^2 \\ &\le \Big(\frac{\pi C'}{\delta}\Big)^2 \sum_{\mathbf{m}\in\mathbb{Z}^d}\big(1 + |\mathbf{m}|\big)^{2(d+\alpha)}e^{-2\rho|\mathbf{m}|} < \infty,\end{aligned}$$

and (28) is proved. □

Now we are able to estimate the spectral projections corresponding to H.

Lemma 10. *For any $f \in \mathcal{H}$ and any $\delta > 0$ one has*

$$\lim_{\varepsilon\to 0+} \varepsilon \int_{\widetilde{\Delta}_\delta} \|(H - \lambda - i\varepsilon)^{-1}f\|^2 d\lambda = 0. \tag{29}$$

Proof. Here we are going to use proposition 2. First note that due to $\widetilde{\Delta}_\delta \subset \mathbb{R} \setminus \operatorname{spec} H_0$ one has

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\widetilde{\Delta}_\delta} \|(H_0 - \lambda - i\varepsilon)^{-1}f\|^2 d\lambda = 0 \text{ for any } f \in \mathcal{H}. \tag{30}$$

Similar to (15) let us consider the decomposition

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \quad \mathcal{H}_0 := \Big(\bigcup_{\Im z\neq 0}\nu(z)\big(\ell^2(\mathbb{Z}^d)\big)\Big)^\perp, \quad \mathcal{H}_1 := \mathcal{H}_0^\perp;$$

As previously, by (9), for any $f \in \mathcal{H}_0$ and any z with $\Im z \ne 0$ one has $\nu^*(z)f = 0$. Hence, by (5), there holds $(H - z)^{-1}f = (H_0 - z)^{-1}$, and (30) implies (29) for $f \in \mathcal{H}_0$.

Now it is sufficient to show (30) for vectors $f = \nu(\zeta)h$ for $h = \big(N(\zeta) - B\big)^{-1}\delta_{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^d$, $\Im\zeta \ne 0$. The operators $\big(N(\zeta) - B\big)^{-1}$ have dense range (coinciding with

$\operatorname{dom} B$), hence the linear hull of such vectors f is dense in $\mathcal{H}_1$. By elementary calculations (see e.g. section 3 in [5]) one rewrites Eq. (5) as

$$(H-\lambda-i\varepsilon)^{-1}f=\frac{1}{\zeta-\lambda-i\varepsilon}\Big(f-\nu(\lambda+i\varepsilon)\big(N(\lambda+i\varepsilon)-B\big)^{-1}\delta_{\mathbf{m}}\Big). \tag{31}$$

Due to lemma 9 we have $\big\|\big(N(\lambda+i\varepsilon)-B\big)^{-1}\delta_{\mathbf{m}}\big\|\le C$ with some $C>0$, for all $\lambda\in\widetilde{\Delta}_\delta$ and sufficiently small ε, and (31) implies

$$\|(H-\lambda-i\varepsilon)^{-1}f\|\le\frac{\|f\|+C\|\nu(\lambda+i\varepsilon)\|}{|\zeta-\lambda-i\varepsilon|},$$

and due to the analyticity of γ, one can estimate $\|(H-\lambda-i\varepsilon)^{-1}f\|\le C'$ with some $C'>0$ for all $\lambda\in\widetilde{\Delta}_\delta$ and sufficiently small ε. This obviously implies (29). $\square$

Proposition 11. *The spectrum of H outside* $\operatorname{spec} L$ *is pure point.*

Proof. We are going to show that for any $f\in\mathcal{H}$ and any interval $\Delta\subset\mathbb{R}\setminus\operatorname{spec} H_0$ the spectral measure μ_f associated with H and f satisfies $\mu_f(\Delta)=\mu_f\big(\Delta\cap\bigcup_m\{\lambda(m)\}\big)$; this proves that all the spectral measures are pure point.

By the Stone formula, for any set X which is a countable union of intervals whose ends are not eigenvalues of H one has

$$\mu_f(X)=\lim_{\varepsilon\to0+}\frac{\varepsilon}{\pi}\int_X\|(H-\lambda-i\varepsilon)f\|^2d\lambda.$$

Using lemma 10, for any $\delta>0$ we estimate

$$\begin{aligned}\mu_f(\Delta)&=\lim_{\varepsilon\to0+}\frac{\varepsilon}{\pi}\int_\Delta\|(H-\lambda-i\varepsilon)f\|^2d\lambda\\&=\lim_{\varepsilon\to0+}\frac{\varepsilon}{\pi}\int_{\Delta_\delta}\|(H-\lambda-i\varepsilon)f\|^2d\lambda+\lim_{\varepsilon\to0+}\frac{\varepsilon}{\pi}\int_{\widetilde{\Delta}_\delta}\|(H-\lambda-i\varepsilon)f\|^2d\lambda\\&=\lim_{\varepsilon\to0+}\frac{\varepsilon}{\pi}\int_{\Delta_\delta}\|(H-\lambda-i\varepsilon)f\|^2d\lambda=\mu_f(\Delta_\delta).\end{aligned}$$

As δ is arbitrary and $\bigcap_{\delta>0}\Delta_\delta=\bigcup_m\{\lambda(m)\}$, the theorem is proved. $\square$

Acknowledgments

The work was supported by the Marie Curie Intra-European Fellowship PIEF-GA-2008-219641 during the stay at the University Paris Nord in July-Septembre 2008.

References

[1] J. Bellissard, R. Lima, E. Scoppola: *Localization in ν-dimensional incommensurate structures.* Commun. Math. Phys. **88** (1983) 465–477.

[2] F. Bentosela, Ph. Briet, L. Pastur: *On the spectral and wave propagation properties of the surface Maryland model.* J. Math. Phys. **44** (2003) 1–35.

[3] F. Bentosela, Ph. Briet, L. Pastur: *Spectral analysis of the generalized surface Maryland model.* St. Petersbg. Math. J. **16** (2005) 923–942.

[4] J. F. Brasche, M. M. Malamud, H. Neidhardt: *Weyl function and spectral properties of self-adjoint extensions.* Integr. Equations Operator Theory **43** (2002) 264–289.

[5] J. Brüning, V. Geyler, K. Pankrashkin: *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators.* Rev. Math. Phys. **20** (2008) 1–70.

[6] P. Exner, M. Fraas: *On the dense point and absolutely continuous spectrum for Hamiltonians with concentric δ shells.* Lett. Math. Phys. **82** (2007) 25–37.

[7] A. L. Figotin, L. A. Pastur: *An exactly solvable model of a multidimensional incommensurate structure.* Commun. Math. Phys. **95** (1984) 401–425.

[8] S. Gnutzmann, U. Smilansky: *Quantum graphs: Applications to quantum chaos and universal spectral statistics.* Adv. Phys. **55** (2006) 527-625.

[9] D. R. Grempel, S. Fishman, R. E. Prange: *Localization in an incommensurate potential: an exactly solvable model.* Phys. Rev. Lett. **49** (1982) 833–836.

[10] B. A. Khoruzhenko, L. A. Pastur, *The localization of surface states: an exactly solvable model.* Phys. Rep. **288** (1997) 109–126.

[11] P. Kuchment: *Quantum graphs* I. *Some basic structures.* Waves Random Media **14** (2004) S107–S128.

[12] P. Kuchment: *Quantum graphs* II. *Some spectral properties of quantum and combinatorial graphs.* J. Phys. A: Math. Gen. **38** (2005) 4887–4900.

[13] K. Pankrashkin: *Spectra of Schrödinger operators on equilateral quantum graphs.* Lett. Math. Phys. **77** (2006) 139–154.

[14] K. Pankrashkin: *Localization in a quasiperiodic model on quantum graphs.* In the book P. Exner et al. (Eds): *Analysis on graphs and its applications* (Proc. Symp. Pure Math., vol. 77, AMS, Provdence, 2008) 459–467.

Laboratoire de Mathématiques, Université Paris Sud, Bâtiment 425, 91405 Orsay Cedex, France

E-mail address: konstantin.pankrashkin@math.u-psud.fr